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Quantal Noether Identities and Their Applications

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Based on the phase-space generating functional of the Green function for a system with a regular/singular Lagrangian, the quantal canonical Noether identities (NI) under the local and non-local transformation in extended phase have been derived, respectively. The result holds true whether the Jacobian of the transformation is equal to unity or not. Based on the configuration-space generating functional of the gauge-invariant system obtained by using Faddeev-Popov (FP) trick, the quantal NI under the local and non-local transformation in configuration space have been also deduced. It is showed that for a system with a singular Lagriangian one must use the effective action in the quantal NI instead of the classical action in corresponding classical NI. It is pointed out that in certain cases, the quantal NI may be converted into the quantal (weak) conservation laws by using the quantal equations of motion. This algorithm to derive the quantal conservation laws differs from the quantal first Noether theorem. The preliminary applications of this formulation to Yang-Mills (YM) fields and non-Abelian Chern-Simons (CS) theories are given. The quantal conserved quantities for non-local transformation in YM fields are obtained. The conserved BRS and PBRS quantities at the quantum level in non-Abelian CS theories are also found. The property of fractional spin in CS theories is discussed.

KEY WORDS: symmetries; noether identities; chern-Simons theories.

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1. INTRODUCTION

Symmetry is now a fundamental concept in modern field theories. In classical theories, the connection between the invariance of the action integral under finite continuous group (global symmetry) and conservation laws is given by the first Noether theorem. The classical second Noether theorem refers to the invariance of an action integral under an infinite continuous group (local symmetry). In this

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case there exist some differential identities which involve variational derivatives of the action integral, and these identities are called Noether identities (NI). They play an important role in field theories (Li, 1993a). Classical Noether identities and their generalization for non-local transformation are usually formulated in terms of Lagrangian variables in configuration space (Li, 1993a, 1995a). Classical Noether theorems in canonical formalism had been established in the previous works (Li, 1991, 1993b, 1994a). These theorems are useful tools for the study of the canonical system with constraints in Dirac's sense, and the properties of Lagrange multipliers connected with the first-class constraints and the invalidity of Dirac's conjecture had been discussed (Li, 1991, 1994a). When we apply them to Yang-Mills theories, the classical NI may be converted into the conservation laws along the trajectory of the motion (Li, 1991, 1993b, 1994a,b). In certain cases, the quantized effective Lagrangian obtained by using Faddeev-Popov(FP) trick is used to derive those conservation laws. Thus, the formulation is a semi-classical theory which is not constructed in a totally quantum theory by making a thorough investigation. Whether those results are valid at the quantum level needs further study. The quantal canonical first Noether theorem had been also formulated in the previous works (Li, 1995b, 1997; Li and Long, 1999). Now the quantal NI for local and non-local transformation will be established, and some applications to YM fields and CS theories will be given.

They are the kind of symmetries that we must consider when dealing with the quantum system, the path integrals provide a useful tool where main ingredient is the classical action together with the measure in the space of field configuration. The phase-space path integrals are more basic than configuration-space path integrals, the latter provide a Hamiltonian quadratic in canonical momenta, whereas the former apply to arbitrary Hamiltonian (Mizrahi, 1978). In certain integrable cases (for example, YM theories), phase-space integral can be simplified by carrying out explicit integration over canonical momenta. Then, the phase-space path integral can be represented in the form of a path integral only over the coordinates (or field variables) of the expression containing a certain Lagrangian (or effective Lagrangian) in configuration space. In more general cases, especially for the constrained Hamiltonian system with complicated constraints, it is very difficult or even impossible to carry out the path integral over the canonical momenta. Thus, the study of symmetry in phase-space path integral formulation has a more fundamental sense. The phase-space path integral formalism makes the symmetries of the system manifest in quantum theories.

Local gauge invariance is a central concept in modern field theories. A system with a gauge-invariant Lagrangian is subject to some inherent phase-space constraints, which is a constrained Hamiltonian system. The path-integral quantization of this system can be formulated with aid of the Dirac theory of constrained system and the method of the path (functional) integration (Batalin and Vilkvisky, 1977, 1983; Faddeev, 1970; Fradkin and Fradkin, 1978; Fradkin and Vilkovisky, 1975;

Gomis *et al.*, 1995; Henneaux, 1985; Senjanovic, 1976). However, for a gauge-invariant system one can conveniently use the FP trick (Faddeev and Popov, 1967) to formulate its path-integral quantization in configuration space. In certain cases, according to the path-integral quantization of the constrained Hamiltonian system, one can carry out explicit integration of canonical momenta in the phase-space path integral which may be coverted to the same results obtained by using the FP trick (for example, YM theories). Although the FP trick is not a rigorous method, it is a simple and more useful method for the gauge theories.

In this paper, based on the phase-space generating functional of the Green function, the canonical NI under the local and non-local transformation at the quantum level have been derived. For the gauge-invariant system, based on configuration-space generating functional obtained by using FP trick, the quantal NI under the local and non-local transformation in configuration space are also deduced. The results hold true no matter whether the Jacobian of the transformation is equal to unity or not. The expressions of quantal NI differ from classical ones for a system with a singular Lagrangian in that one must use quantized effective action instead of classical action in corresponding expressions. It is pointed out that in certain cases based on the quantal NI, one can obtain quantal conservation laws of the system, this algorithm to derive quantal conservation laws makes a thorough study in quantum theory which is totally different from quantal first Noether theorem. Finally, we give some applications of above results to the YM fields and CS theories.

The paper is organized as follows. In Section 2, the quantal canonical NI under the local and non-local transformation in phase space have been derived. These identities coincide with classical ones for a regular Lagrangian, but for a singular Lagrangian one must use I_{eff}^{P} instead of I^{P} in those identities. In Section 3, based on quantal canonical Noether identities, in a certain case the existence of strong and weak conserved laws have been discussed. In Section 4, quantal NI under the local and non-local transformation in configuration space for a gauge-invariant system have been deduced, and the quantal conserved laws connected with these identities are also discussed. The applications of above formulation to YM fields are given in Section 5, some quantal conserved quantities for local and non-local transformation are obtained. In Section 6, we give some applications to non-Abelian CS theories with Maxwell term, the quantal BRS and PBRS conserved quantity and quantal conserved angular momentum are obtained, and property of fractional spin at quantum level for non-Abelian CS theories needs further study. Section 7 is devoted to conclusions and discussion.

2. QUANTAL CANONICAL NOETHER IDENTITIES

Let us first consider a physical field defined by the field variable $\varphi(x)$ ($\varphi(x)$ represents all field variables) and the motion of field described by a regular Lagrangian density $\mathsf{L}(\varphi, \varphi_{,\mu})$, $\varphi_{,\mu} = \partial_\mu \varphi = \partial\varphi/\partial x^\mu$, where $x = (t, \vec{x})$. The flat space-time metric is $g_{\mu\nu} = (1\ -1\ -1\ -1)$. The canonical Hamiltonian $H_C = \int d^3x \mathsf{H}_C = \int d^3x(\pi\dot{\varphi} - \mathsf{L})$ is a functional of independent canonical variables $\varphi(x)$ and $\pi(x)$, where $\pi(x) = \partial\,\mathsf{L}/\partial\dot{\varphi}(x)$ is a canonical momenta conjugating to $\varphi(x)$, H_C is a canonical Hamiltonian density. We adopt the path-integral quantization for the system. The phase-space generating functional of the Green function in the form of a path (functional) integral is (Li, 1994c)

$$Z[J, K] = \int \mathsf{D}\varphi\, \mathsf{D}\pi \exp\left\{i \int d^4x(\mathsf{L}^P + J\varphi + K\pi)\right\} \tag{2.1}$$

where

$$\mathsf{L}^P = \pi\dot{\varphi} - \mathsf{H}_C \tag{2.2}$$

and J, K are the exterior sources with respect to φ and π respectively. Here we have also introduced the exterior source K with respect to canonical momenta π, which does not alter the calculation of the Green function G

$$G(x_1, x_2, \ldots, x_n) = \frac{1}{i^n} \quad \frac{\delta^n Z[J, K]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\bigg|_{J=K=0} \tag{2.3}$$

Based on the phase-space generating functional, the canonical first Noether theorem at the quantum level have been established for global symmetries (Li, 1995b, 1997; Li and Long, 1999). Now we further discuss local and non-local transformation. Local gauge invariance is a basic concept in modern field theories, and non-local transformations in field theories also have been introduced (Fradkin and Palchik, 1984; Kuang and Yi, 1980; Li and Long, 1999; Rabello and Gaete, 1995). Let us consider the transformation properties of the system under the local and non-local transformation in extended phase space, whose infinitesimal transformation is given by

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + R^\mu_\sigma \varepsilon^\sigma(x) \\ \varphi'(x') = \varphi(x) + \Delta\varphi(x) = \varphi(x) + S_\sigma\varepsilon^\sigma(x) + \int d^4x E(x, y) A_\sigma(y)\varepsilon^\sigma(y) \\ \pi'(x') = \pi(x) + \Delta\pi(x) = \pi(x) + T_\sigma\varepsilon^\sigma(x) + \int d^4x F(x, y) B_\sigma(y)\varepsilon^\sigma(y) \end{cases} \tag{2.4}$$

where $E(x, y)$ and $F(x, y)$ are some functions, R^μ_σ, S_σ, T_σ, A_σ and B_σ are linear differential operators, for example,

$$R^\mu_\sigma = r^{\mu\nu\ldots\lambda}_\sigma \partial_\nu, \ldots, \partial_\lambda, \text{etc.} \tag{2.5}$$

where the summation over the repeat indices is taken, and $r^{\mu\nu\ldots\lambda}_\sigma$ are functions of x, φ and π, and $\varepsilon^\sigma(x)$ ($\sigma = 1, 2, \ldots, r$) are arbitrary infinitesimal functions, and their values and derivatives up to required order vanish on the boundary of the space-time domain. The Jacobian of the transformation of the canonical variables defined by (2.4) is denoted by $\bar{J}[\varphi, \pi, \varepsilon] = 1 + J_1[\varphi, \pi, \varepsilon]$, where J_1 is also an infinitesimal quantity. It is supposed that the variation of the canonical action integral under the transformation (2.4) is given by

$$\Delta I^P = \Delta \int d^4x\, \mathsf{L}^P = \int d^4x U_\sigma \varepsilon^\sigma(x) \tag{2.6}$$

where U_σ are also linear differential operators. Under the transformation (2.4), from the expression (2.1) of the phase-space generating functional and (2.6), one obtains

$$\begin{aligned} &\int \mathsf{D}\varphi\, \mathsf{D}\pi (1 + J_1 + i\Delta I^P + i \int d^4x \{J\delta\varphi + K\delta\pi + \partial_\mu[(J\varphi + K\pi)\Delta x^\mu]\}) \\ &\quad \times \exp\{i \int d^4x(\mathsf{L}^P + J\varphi + K\pi)\} \qquad (2.7) \\ &= \int \mathsf{D}\varphi \mathsf{D}\pi (1 + J_1 + i \int d^4x\{U_\sigma\varepsilon^\sigma(x) + J\delta\varphi + K\delta\pi + \partial_\mu[(J\varphi + K\pi)\Delta x^\mu]\}) \\ &\quad \times \exp\{i \int d^4x(\mathsf{L}^P + J\varphi + K\pi)\} \end{aligned}$$

where (Li, 1993a)

$$\Delta I^P = \int d^4x \left\{\frac{\delta I^P}{\delta\varphi}\delta\varphi + \frac{\delta I^P}{\delta\pi}\delta\pi + D(\pi\delta\varphi) + \partial_\mu[(\pi\dot{\varphi} - \mathsf{H}_C)\Delta x^\mu]\right\} \tag{2.8}$$

$$\frac{\delta I^P}{\delta\varphi} = -\dot{\pi} - \frac{\delta H_C}{\delta\varphi}, \qquad \frac{\delta I^P}{\delta\pi} = \dot{\varphi} - \frac{\delta H_C}{\delta\pi} \tag{2.9}$$

$$\delta\varphi = \Delta\varphi - \varphi_{,\mu}\Delta x^\mu, \qquad \delta\pi = \Delta\pi - \pi_{,\mu}\Delta x^\mu \tag{2.10}$$

where $D = d/dt$. According to the boundary condition of the functions $\varepsilon^\sigma(x)$, from (2.7) and (2.8), one gets

$$\begin{aligned} &\int D\varphi D\pi \left[\frac{\delta I^P}{\delta\varphi}\delta\varphi + \frac{\delta I^P}{\delta\pi}\delta\pi + D(\pi\delta\varphi) - U_\sigma\varepsilon^\sigma(x)\right] \\ &\quad \exp\left\{i \int d^4x(\mathsf{L}^P + J\varphi + K\pi)\right\} = 0 \end{aligned} \tag{2.11}$$

We substitute (2.4) and (2.10) into (2.11), and integrate by parts for corresponding terms, then functionally differentiate the results with respect to $\varepsilon^\sigma(x)$, according to the boundary condition of the functions $\varepsilon^\sigma(x)$, we obtain

$$\begin{aligned} &\int \mathsf{D}\varphi\, \mathsf{D}\pi \left(\tilde{S}_\sigma(x)\left(\frac{\delta I^P}{\delta\varphi(x)}\right) + \tilde{T}_\sigma(x)\left(\frac{\delta I^P}{\delta\pi(x)}\right)\right. \\ &\quad - \tilde{R}^\mu_\sigma(x)\left[\varphi_{,\mu}(x)\frac{\delta I^P}{\delta\varphi(x)} + \pi_{,\mu}(x)\frac{\delta I^P}{\delta\pi(x)}\right] \end{aligned}$$

$$
\begin{aligned}
&+\int d^4z\left\{\tilde{A}_\sigma(z)\left[E(z,x)\frac{\delta I^P}{\delta\varphi(z)}+D(\pi(z)E(z,x))\right]\right.\\
&\left.+\;\tilde{B}_\sigma(z)\left(F(z,x)\frac{\delta I^P}{\delta\pi(z)}\right)\right\}-\tilde{U}_\sigma(1)\Bigg)\\
&\times\exp\left\{i\int d^4x(\mathsf{L}^P+J\varphi+K\pi)\right\}=0
\end{aligned}
\tag{2.12}
$$

where $\tilde{S}_\sigma, \tilde{T}_\sigma, \tilde{R}^\mu_\sigma, \tilde{A}_\sigma, \tilde{B}_\sigma$ and $\tilde{U}_\sigma$ are adjoint operators with respect to $S_\sigma, T_\sigma, R^\mu_\sigma, A_\sigma, B_\sigma$ and U_σ respectively (Li, 1987). For example, $\int fR^\mu_\sigma g d^4x = \int g\tilde{R}^\mu_\sigma f d^4x + [\cdot]_B$, where $[\cdot]_B$ stands for boundary terms.

Functionally differentiating (2.12) with respect to $J(x)$ a total of n times, one obtains

$$
\begin{aligned}
&\int \mathsf{D}\varphi\,\mathsf{D}\pi\left(\tilde{S}_\sigma(x)\left(\frac{\delta I^P}{\delta\varphi(x)}\right)+\tilde{T}_\sigma(x)\left(\frac{\delta I^P}{\delta\pi(x)}\right)\right.\\
&-\tilde{R}^\mu_\sigma(x)\left[\varphi_{,\mu}(x)\frac{\delta I^P}{\delta\varphi(x)}+\pi_{,\mu}(x)\frac{\delta I^P}{\delta\pi(x)}\right]\\
&+\int d^4z\left\{\tilde{A}_\sigma(z)\left[E(z,x)\frac{\delta I^P}{\delta\varphi(z)}+D\left(\pi(z)E(z,x)\right)\right]\right.\\
&\left.+\;\tilde{B}_\sigma(z)\left(F(z,x)\frac{\delta I^P}{\delta\pi(z)}\right)\right\}-\tilde{U}_\sigma(1)\Bigg)\\
&\times\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\exp\left\{i\int d^4x(\mathsf{L}^P+J\varphi+K\pi)\right\}=0
\end{aligned}
\tag{2.13}
$$

Let $J = K = 0$ in (2.13), one gets

$$
\begin{aligned}
&\langle 0\big|T^*\Bigg(\tilde{S}_\sigma(x)\left(\frac{\delta I^P}{\delta\varphi(x)}\right)+\tilde{T}_\sigma(x)\left(\frac{\delta I^P}{\delta\pi(x)}\right)\\
&-\tilde{R}^\mu_\sigma(x)\left[\varphi_{,\mu}(x)\frac{\delta I^P}{\delta\varphi(x)}+\pi_{,\mu}(x)\frac{\delta I^P}{\delta\pi(x)}\right]\\
&+\int d^4z\{\tilde{A}_\sigma(z)\left[E(z,x)\frac{\delta I^P}{\delta\varphi(z)}+D\Big(\pi(z)E(z,x)\Big)\right]\\
&+\tilde{B}_\sigma(z)\left(F(z,x)\frac{\delta I^P}{\delta\pi(z)}\right)-\tilde{U}_\sigma(1)\Bigg)\\
&\cdot\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n))\big|0\Big\rangle=0
\end{aligned}
\tag{2.14}
$$

where the symbol T^* stands for covariantized T product (Young, 1987), in which derivatives of operators inside a T-product are defined in terms of the formula, i.e.

$$\langle 0|T^*[\partial_\mu\varphi(x)\partial_\nu\varphi(y)\ldots]|0\rangle = \partial_\mu\partial_\nu\,\langle 0|T[\varphi(x)\varphi(y)\ldots]|0\rangle$$

and $|0\rangle$ is the vacuum state of the fields. Fixing t and letting $t_1, t_2, \ldots, t_m \to +\infty$, $t_{m+1}, t_{m+2}, \ldots, t_n \to -\infty$, noting $\varphi(\vec{x}, -\infty)|0\rangle = |\text{in}\rangle$, $\varphi(\vec{x}, \infty)|0\rangle = \langle\text{out}|$, and using the reduction formula (Young, 1987), we can write expression (2.14) as

$$\begin{aligned}
&\Big\langle \text{out}, m\Big|\left(\tilde{S}_\sigma(x)\left(\frac{\delta I^P}{\delta\varphi(x)}\right) + \tilde{T}_\sigma(x)\left(\frac{\delta I^P}{\delta\pi(x)}\right)\right.\\
&-\tilde{R}^\mu_\sigma(x)\left[\varphi_{,\mu}(x)\frac{\delta I^P}{\delta\varphi(x)} + \pi_{,\mu}(x)\frac{\delta I^P}{\delta\pi(x)}\right]\\
&+\int d^4z\left\{\tilde{A}_\sigma(z)\left[E(z,x)\frac{\delta I^P}{\delta\varphi(z)} + D\Big(\pi(z)E(z,x)\Big)\right]\right\}\\
&\left.+\tilde{B}_\sigma(z)\left(F(z,x)\frac{\delta I^P}{\delta\pi(z)}\right) - \tilde{U}_\sigma(1)\right)\Big|n-m, in\Big\rangle = 0
\end{aligned} \tag{2.15}$$

Since m and n are arbitrary, one obtains

$$\begin{aligned}
&\tilde{S}_\sigma(x)\left(\frac{\delta I^P}{\delta\varphi(x)}\right) + \tilde{T}_\sigma(x)\left(\frac{\delta I^P}{\delta\pi(x)}\right) - \tilde{R}^\mu_\sigma(x)\left(\varphi_{,\mu}(x)\frac{\delta I^P}{\delta\varphi(x)} + \pi_{,\mu}(x)\frac{\delta I^P}{\delta\pi(x)}\right)\\
&+\int d^4z\left\{\tilde{A}_\sigma(z)\left[E(z,x)\frac{\delta I^P}{\delta\varphi(z)} + D\Big(\pi(z)E(z,x)\Big)\right]\right.\\
&\left.+\tilde{B}_\sigma(z)\left(F(z,x)\frac{\delta I^P}{\delta\pi(z)}\right)\right\} - \tilde{U}_\sigma(1) = 0
\end{aligned} \tag{2.16}$$

These expressions are called quantal canonical NI under the local and nonlocal transformation (2.4) for a system with a regular Lagrangian. For the case $E = F = 0$, the transformation (2.4) will be converted into a local one, and (2.16) can be written as

$$\begin{aligned}
&\tilde{S}_\sigma(x)\left(\frac{\delta I^P}{\delta\varphi(x)}\right) + \tilde{T}_\sigma(x)\left(\frac{\delta I^P}{\delta\pi(x)}\right) - \tilde{R}^\mu_\sigma(x)\left[\varphi_{,\mu}(x)\frac{\delta I^P}{\delta\varphi(x)} + \pi_{,\mu}(x)\frac{\delta I^P}{\delta\pi(x)}\right]\\
&-\tilde{U}_\sigma(1) = 0
\end{aligned} \tag{2.17}$$

This expresions coincide with the classical NI (Li, 1993c) whether the Jacobian of the transformation (2.4) is equal to unity or not.

Let us now consider a system with a Singular Lagrangian $\mathsf{L}(\varphi^\alpha, \varphi^\alpha_{,\mu})$ whose Hessian matrix$[H_{\alpha\beta}] = [\partial^2\mathsf{L}/\partial\dot{\varphi}^\alpha\partial\dot{\varphi}^\beta]$ is degenerate. Using the Legendre transformation, one can go over from the Lagrangian description to the Hamiltonian description, and the motion of the system is described by the canonical variables, which is subject to some inherent phase-space constraints and is called constrained Hamiltonian system. Let $\Lambda_k(\varphi^\alpha, \pi_\alpha) \approx 0$ $(k = 1, 2, \ldots, K_1)$ be first-class constraints, and $\theta_i(\varphi^\alpha, \pi_\alpha) \approx 0$ $(i = 1, 2, \ldots, I_1)$ be second-class constraints. The path-integral quantization for this system can be formulated by using Batalin–Fradkin–Vilkovisky (BFV) scheme (Batalin and Vilkvisky, 1977; Fradkin and Fradkin, 1978; Fradkin and Vilkovisky, 1975; Henneaux, 1985), or Batalin-Vilkovisky (BV) scheme (Batalin and Vilkovisky, 1983; Gomis *et al.*, 1995), or Faddeev–Senjanovic (FS) scheme (Faddeev, 1970; Senjanovic, 1976), but the latter is more convenient. According to FS path-integral quantization scheme, the gauge conditions connecting with the first-class constraints can be chosen as $\Omega_i(\varphi^\alpha, \pi_\alpha) \approx 0$ $(k = 1, 2, \ldots, K_1)$, the phase-space generating functional of the Green function for this constrained Hamiltonian system can be written as

$$\begin{aligned} Z[J, K] = \int \mathrm{D}\varphi^\alpha \mathrm{D}\pi_\alpha \prod_{i,k,l} \delta(\theta_i)\delta(\Lambda_k)\delta(\Omega_l) \det|\{\Lambda_k,\ \Omega_l\}| \cdot \\ [\det|\{\theta_i, \theta_j\}|]^{1/2} \exp\left\{i \int d^4x (\mathrm{L}^P + J_\alpha\varphi^\alpha + K^\alpha\pi_\alpha)\right\} \end{aligned} \tag{2.18}$$

where $\{\cdot, \cdot\}$ represents Poission bracket, J_α and K^α are the exterior sources with respect to φ^α and K_α, respectively (Li, 1994c). Using the δ-function and integral properties of the Grassmann variables $C_a(x)$ and $\bar{C}_b(x)$, one can write (2.18) as (Li, 1994c)

$$\begin{aligned} Z[J, K, \eta^m, \bar{j}, \bar{k}, j, k] = \int \mathrm{D}\varphi^\alpha \mathrm{D}\pi_\alpha \mathrm{D}\lambda_m \mathrm{D}\,\bar{C}_a \mathrm{D}\pi^a \mathrm{D}\,C_a \mathrm{D}\bar{\pi}^a \\ \cdot \exp\left\{i \int d^4x \left(\mathrm{L}_{\mathrm{eff}}^P + J_\alpha\varphi^\alpha + K^\alpha\pi_\alpha + \eta^m\lambda_m + \bar{j}^a C_a + \bar{k}_a\pi^a + \bar{C}_a j^a + \bar{\pi}^a k_a\right)\right\} \end{aligned} \tag{2.19}$$

where

$$\mathrm{L}_{\mathrm{eff}}^P = \mathrm{L}^P + \mathrm{L}_m + \mathrm{L}_{gh} \tag{2.20}$$

$$\mathrm{L}_m = \lambda_k\Lambda_k + \lambda_l\Omega_l + \lambda_i\theta_i \tag{2.21}$$

$$L_{gh} = \int d^4y \left[\bar{C}_k(x)\{\Lambda_k(x), \Omega_l(y)\}C_l(y) + \frac{1}{2}\bar{C}_i(x)\{\theta_i(x), \theta_j(y)\}\theta_j(y)\right] \tag{2.22}$$

and $\lambda_m = (\lambda_k, \lambda_l, \lambda_i)$, $\bar{\pi}^a(x)$ and $\pi^b(x)$ are canonical momenta conjugate to $C_a(x)$ and $\bar{C}_b(x)$, respectively. η^m, $\bar{j}^a$, $\bar{k}_a$, j^a and k_a are exterior sources with respect to $\lambda^m, C_a, \pi^a, \bar{C}^a$ and $\bar{\pi}^a$ respectively, and $\mathrm{L}_{\mathrm{eff}}^P$ is called a quantized effective canonical Lagrangian density. For the sake of simplicity, let us denote $\varphi = (\varphi^\alpha, \lambda_m, C_a, \bar{C}_a)$, $\pi = (\pi_\alpha, \bar{\pi}^a, \pi^a)$, $J = (J_\alpha, \eta^m, j^a, \bar{j}^a)$, and $K = (K^\alpha, k_a, \bar{k}_a)$, thus, the expression (2.18) can be written as

$$Z[J, K] = \int \mathrm{D}\varphi\, \mathrm{D}\pi\ \exp\left\{i \int d^4x\left(\mathrm{L}_{\mathrm{eff}}^P + J\varphi + K\pi\right)\right\} \tag{2.23}$$

For a system with a singular Lagrangian, one can still proceed in the same way as for a system with a regular Lagrangian to deduce the quantal canonical NI under the local and nonlocal transformation in phase space, but in this case one must use I^P_{eff} instead of I^P in the expressions (2.6)–(2.9) and (2.11)–(2.17). In classical theories, the canonical NI for singular Lagrangian are coincide with those ones for regular Lagrangian. But, for a system with a singular Lagrangian, the quantal canonical NI have the same form as expression (2.16) or (2.17) in which one must use I^P_{eff} instead of I^P in (2.16) and (2.17).

Thus, we have identity relations (2.16) and (2.17) between the functional derivatives and their derivatives, and this leads to a reduction in the number of independent functional derivatives $\delta I^P_{\text{eff}}/\delta\varphi$ and $\delta I^P_{\text{eff}}/\delta\pi$.

3. QUANTAL CONSERVATION LAWS

Using the quantal canonical NI, for certain cases one can obtain quantal strong conservation laws which hold true no matter whether the equations of motion at the quantum level are satisfied. Using the quantal equations of motion of the system, one can obtain quantal weak conservation laws. In order to study the applications of the quantal canonical NI to the YM fields and CS theories, we consider the following infinitesimal local transformation

$$\begin{cases} \Delta x^\mu = 0 \\ \delta\varphi(x) = b_\sigma \varepsilon^\sigma(x) + b^\mu_\sigma \partial_\mu \varepsilon^\sigma(x) \\ \delta\pi(x) = c_\sigma \varepsilon^\sigma(x) + c^\mu_\sigma \partial_\mu \varepsilon^\sigma(x) \end{cases} \tag{3.1}$$

where $b_\sigma, b^\mu_\sigma, c_\sigma$ and c^μ_σ are smoothed functions of x, φ and π, and $\varepsilon^\sigma(x)$ ($\sigma = 1, 2, \ldots, r$) are arbitrary infinitesimal functions. It is supposed that the change of the effective canonical Lagrangian density $\mathsf{L}^P_{\text{eff}}$ is given by

$$\delta L^P_{\text{eff}} = u_\sigma \varepsilon^\sigma(x) = \left(u_\sigma + u^\mu_\sigma \partial_\mu + u^{\mu\nu}_\sigma \partial_\mu \partial_\nu\right)\varepsilon^\sigma(x) \tag{3.2}$$

under the transformation (3.1) where u_σ, u^μ_σ and $u^{\mu\nu}_\sigma$ are some functions of x, φ and π. For example, some models in the massive Yang–Mills theories belong to this category. The quantal canonical NI (2.17) in this case becomes

$$b_\sigma \frac{\delta I^P_{\text{eff}}}{\delta\varphi} - \partial_\mu\left(b^\mu_\sigma \frac{\delta I^P_{\text{eff}}}{\delta\varphi}\right) + c_\sigma \frac{\delta I^P_{\text{eff}}}{\delta\pi} - \partial_\mu\left(c^\mu_\sigma \frac{\delta I^P_{\text{eff}}}{\delta\pi}\right) = u_\sigma - \partial_\mu u^\mu_\sigma + \partial_\mu \partial_\nu u^{\mu\nu}_\sigma \tag{3.3}$$

From the variation of the effective canonical action under the transformation (3.1), one has

$$\frac{\delta I^P_{\text{eff}}}{\delta\varphi}\left(b_\sigma + b^\mu_\sigma \partial_\mu\right)\varepsilon^\sigma(x) + \frac{\delta I^P_{\text{eff}}}{\delta\pi}\left(c_\sigma + c^\mu_\sigma \partial_\mu\right)\varepsilon^\sigma(x) + \frac{d}{dt}\left[\pi\left(b_\sigma + b^\mu_\sigma \partial_\mu\right)\varepsilon^\sigma(x)\right] = \left(u_\sigma + u^\mu_\sigma \partial_\mu + u^{\mu\nu}_\sigma \partial_\mu \partial_\nu\right)\varepsilon^\sigma(x) \tag{3.4}$$

Multiplying identities (3.3) by $\varepsilon^\sigma(x)$ and summing up with index σ from 1 to r and substracting the result from the basic identity (3.4), if the indices μ, ν of the coefficients $u_\sigma^{\mu\nu}$ are symmetrical, then one obtains

$$\partial_\mu \left[\left(b_\sigma^\mu \frac{\delta I_{\text{eff}}^P}{\delta \varphi} + c_\sigma^\mu \frac{\delta I_{\text{eff}}^P}{\delta \pi} - u_\sigma^\mu + \partial_\nu u_\sigma^{\mu\nu} - u_\sigma^{\mu\nu} \partial_\nu \right) \varepsilon^\sigma(x) \right] + \frac{d}{dt} \left[\pi \left(b_\sigma + b_\sigma^\mu \partial_\mu \right) \varepsilon^\sigma(x) \right] = 0 \tag{3.5}$$

Taking the integral of the identity (3.5) on $t =$ const like-space hepersurface, one gets the strong conservation law:

$$Q = \int_v j_\sigma \varepsilon^\sigma(x) d^3x = \text{const} \tag{3.6}$$

where

$$j_\sigma = b_\sigma^0 \frac{\delta I_{\text{eff}}^P}{\delta \varphi} + c_\sigma^0 \frac{\delta I_{\text{eff}}^P}{\delta \pi} - u_\sigma^0 + \partial_\nu u_\sigma^{0\nu} - u_\sigma^{0\nu} \partial_\nu + \pi \left(b_\sigma + b_\sigma^\mu \partial_\mu \right) \tag{3.7}$$

This conservation law is independent of whether the φ and π are a solution of the quantal canonical equations of the constrained Hamiltonian system.

If the transformation group has a subgroup and $\varepsilon^\sigma(x) = \varepsilon_0^\rho \xi_\rho^\sigma(x)$, where $\varepsilon_0^\rho (\rho = 1, 2, \ldots, s)$ are numerical parameters of the Lie group, and $\xi_\rho^\sigma(x)$ are some functions. For example, BRS transformation in YM theories and the transformation in the discussion of gauge-invariant energy-momentum tensor belong to this category. In this circumstances, the strong conservation law (3.6) becomes:

$$Q_\rho = \int_v j_\sigma \xi_\rho^\sigma d^3x = \text{const} \quad (\rho = 1, 2, \ldots, s) \tag{3.8}$$

Using the quantal canonical equations of the motion, one has (Li, 1997; Li and Long, 1999) $\delta I_{\text{eff}}^P / \delta \varphi = 0$, $\delta I_{\text{eff}}^P / \delta \pi = 0$. From the expression (3.8), one can get the (weak) conservation laws at the quantum level. If the effective canonical action is invariant under the corresponding transformation, then, these quantal conservation laws coincide with the results deriving from the quantal canonical first Noether theorem for the global symmetry transformation in phase space (Li, 1997). Thus, we have seen that the quantal canonical NI may be converted into quantal (weak) conservation laws in certain cases even if the effective canonical action of the system is not invariant under the specific local transformation. This algorithm deriving quantal conservation laws makes a thorough study in quantum theory which differs from the canonical first Noether theorem at the quantum level (Li, 1997).

4. GAUGE-INVARIANT SYSTEM

As is well known, a gauge-invariant system is a constrained Hamiltonian sysetm (Li, 1993a). The quantization of such a system can be formulated by using

FP scheme, the effective Lagrangian L_{eff} in configuration space can be found by using the FP trick through a transformation of the path (functional) integral (Faddeev and Popov, 1967), $\mathsf{L}_{\text{eff}} = \mathsf{L} + \mathsf{L}_f + \mathsf{L}_{gh}$, where L is a gauge-invariant Lagrangian, L_f is determined by the gauge conditions and L_{gh} is a ghost term. The configuration-space generating functional of the Green function for this system can be written as

$$Z[J] = \int \mathsf{D}\varphi \exp\left\{i \int d^4x(\mathsf{L}_{\text{eff}} + J\varphi)\right\} \tag{4.1}$$

where φ represents all field variables, and J is a exterior source with respect to φ. For some models in field theories, the expression (4.1) can be obtained by carrying out explicit integration over canonical momenta in phase-space generating functional for the constrained Hamiltonian system (for example, YM theories).

Let us now consider the transformation properties of the configuration-space generating functional under general local and non-local transformation, whose infinitesimal transformation is given by

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + R^\mu_\sigma \varepsilon^\sigma(x) \\ \varphi'(x') = \varphi(x) + \Delta\varphi(x) = \varphi(x) + S_\sigma \varepsilon^\sigma(x) + \int d^4y E(x, y) N_\sigma(y) \varepsilon^\sigma(y) \end{cases} \tag{4.2}$$

where $\varepsilon^\sigma(x)$ ($\sigma = 1, 2, \ldots, r$) are arbitrary infinitesimal independent functions, the values of $\varepsilon^\sigma(x)$ and their derivatives up to required order on the boundary of space-time domain vanish, and R^μ_σ, S_σ and N_σ are linear differential operators. Under the transformation (4.2), it is supposed that the variation of the effective action is given by

$$\Delta I_{\text{eff}} = \Delta \int d^4x L_{\text{eff}} = \int d^4x V_\sigma \varepsilon^6(x) \tag{4.3}$$

where V_σ are some linear differential operators. The Jacobian of the transformation (4.2) is denoted by $\overline{J} = I + J_1[\varphi, \varepsilon]$. Under the transformation (4.2), the generating functional (4.1) becomes

$$\begin{aligned} Z[J, \varepsilon] = &\int \mathsf{D}\varphi\{I + J_1 + i\Delta I_{\text{eff}} + i \int d^4x[J\delta\varphi + \partial_\mu(J\varphi\Delta x^\mu)]\} \\ &\times \exp\{i \int d^4x(\mathsf{L}_{\text{eff}} + J\varphi)\} \end{aligned} \tag{4.4}$$

where

$$\Delta I_{\text{eff}} = \int d^4x \left[\frac{\delta I_{\text{eff}}}{\delta\varphi}\delta\varphi + \partial_\mu\left(\frac{\partial\,\mathsf{L}_{\text{eff}}}{\partial\varphi'_\mu}\delta\varphi\right) + \partial_\mu(\mathsf{L}_{\text{eff}}\Delta x^\mu)\right] \tag{4.5}$$

$$\frac{\delta I_{\text{eff}}}{\delta\varphi} = \frac{\partial\,\mathsf{L}_{\text{eff}}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\,\mathsf{L}_{\text{eff}}}{\partial\varphi'_\mu}\right) \tag{4.6}$$

$$\delta\varphi = \Delta\varphi - \varphi_{,\mu}\Delta x^{\mu} \tag{4.7}$$

According to the boundary conditions of the $\varepsilon^{\sigma}(x)$, from (4.2)–(4.7), one obtains

$$\int \mathsf{D}\varphi \left\{ \frac{\delta I_{\text{eff}}}{\delta\varphi} \left[\left(S_{\sigma} - \varphi_{,\mu} R_{\sigma}^{\mu}\right)\varepsilon^{\sigma}(x) + \int d^4 y E(x, y) N_{\sigma}(y)\varepsilon^{\sigma}(y) \right] \right.$$
$$\left. + \partial_{\mu} \left[\frac{\partial \mathsf{L}_{\text{eff}}}{\partial \varphi_{,\mu}} \int d^4 y E(x, y) N_{\sigma}(y)\varepsilon^{\sigma}(y) \right] - V_{\sigma}\varepsilon^{\sigma}(x) \right\}$$
$$\times \exp\left\{ i \int d^4 x [\mathsf{L}_{\text{eff}} + J\varphi) \right\} = 0 \tag{4.8}$$

One repeats the integration by part of the terms concerning the differential operators S_{σ}, R_{σ}^{μ}, $A_{\sigma}$and V_{σ} in expression (4.8), appealing to the arbitrariness of the $\varepsilon^{\sigma}(x)$, one can force the boundary terms to vanish. After this one can functionally differentiate the obtained result with respect to $\varepsilon^{\sigma}(x)$, one gets

$$\int \mathsf{D}\varphi \left\{ \tilde{S}_{\sigma} \left(\frac{\delta I_{\text{eff}}}{\delta\varphi(x)} \right) - \tilde{R}_{\sigma}^{\mu} \left(\varphi_{,\mu}(x) \frac{\delta I_{\text{eff}}}{\delta\varphi(x)} \right) + \int d^4 y \tilde{N}_{\sigma} \left[E(y, x) \frac{\delta I_{\text{eff}}}{\delta\varphi(y)} \right. \right.$$
$$\left. \left. + \partial_{\mu} \left(E(y, x) \frac{\partial \mathsf{L}_{\text{eff}}}{\partial \varphi_{,\mu}(y)} \right) \right] - \tilde{V}_{\sigma}(1) \right\} \exp\{ i \int d^4 x (\mathsf{L}_{\text{eff}} + J\varphi) \} = 0 \tag{4.9}$$

where $\tilde{S}_{\sigma}$, $\tilde{R}_{\sigma}^{\mu}$, $\tilde{A}_{\sigma}$ and $\tilde{V}_{\sigma}$ are the adjoint operators with respect to S_{σ}, R_{σ}^{μ}, A_{σ} and V_{σ}, respectively (Li, 1987). Functionally differentiating (3.9) with respect to $J(x)$ n times, one can proceed the same way as discussed in Section 2 to obtain

$$\tilde{S}_{\sigma} \left(\frac{\delta I_{\text{eff}}}{\delta\varphi(x)} \right) - \tilde{R}_{\sigma}^{\mu} \left(\varphi_{,\mu}(x) \frac{\delta I_{\text{eff}}}{\delta\varphi(x)} \right)$$
$$+ \int d^4 y \tilde{N}_{\sigma} \left[E(y, x) \frac{\delta I_{\text{eff}}}{\delta\varphi(y)} + \partial_{\mu} \left(E(y, x) \frac{\partial\, \mathsf{L}_{\text{eff}}}{\partial \varphi_{,\mu}(y)} \right) \right]$$
$$- \tilde{V}_{\sigma}(1) = 0 \quad (\sigma = 1, 2, \ldots, r) \tag{4.10}$$

The expression (4.10) are called quantal NI in configuration space for gauge-invariant system under the local and non-local transformation. For the local transformation ($E = 0$ in (4.2)), from (4.10) one has

$$\tilde{S}_{\sigma} \left(\frac{\delta I_{\text{eff}}}{\delta\varphi} \right) - \tilde{R}_{\sigma}^{\mu} \left(\varphi_{,\mu} \frac{\delta I_{\text{eff}}}{\delta\varphi} \right) - \tilde{V}_{\sigma}(1) = 0 \tag{4.11}$$

The identities (4.11) differ from classical ones in that the action in quantial NI is an effective action I_{eff}, but not a classical one I.

Now, let us consider following infinitesimal local transformation

$$\begin{cases} \Delta x^{\mu} = 0 \\ \delta\varphi(x) = \left(b_{\sigma} + b_{\sigma}^{\mu}\partial_{\mu}\right)\varepsilon^{\sigma}(x) \end{cases} \tag{4.12}$$

where $\varepsilon^{\sigma}(x)$ $(\sigma = 1, 2, \ldots, r)$ are arbitrary infinitesimal functions. It is supposed that the change of the effective Lagrangian L_{eff} is given by

$$\delta \mathsf{L}_{\text{eff}} = V_{\sigma}\varepsilon^{\sigma}(x) = \left(v_{\sigma} + v_{\sigma}^{\mu} + v_{\sigma}^{\mu\nu}\partial_{\mu}\partial_{\nu}\right)\varepsilon^{\sigma}(x) \tag{4.13}$$

under the transformation (4.12), where v_{σ}, v_{σ}^{μ} and $v_{\sigma}^{\mu\nu}$ are some functions of x, φ and $\varphi_{'\mu}$. From the quantal NI (4.11) and the variation of an effective action I_{eff}, one can also deduce the strong conservation laws as did in the Section 3. If $\varepsilon^{\sigma}(x) = \varepsilon_0^{\rho}\zeta_{\rho}^{\sigma}(x)$ in the case, the strong conservation laws become

$$Q_{\sigma} = \int_V d^3x \left[\frac{\partial\, \mathsf{L}_{\text{eff}}}{\partial\varphi_{'0}}\left(b_{\rho} + b_{\rho}^{\nu}\partial_{\nu}\right) + b_{\rho}^{0}\frac{\delta I_{\text{eff}}}{\delta\varphi} - v_{\rho}^{0} + \partial_{\nu}v_{\rho}^{0\nu} - v_{\rho}^{0\nu}\partial_{\nu}\right]\zeta_{\sigma}^{\rho} = \text{const} \tag{4.14}$$

Using the quantal equations of the motion of the system (Li, 1993a; Young, 1987), $\delta I_{\text{eff}}/\delta\varphi = 0$, from (4.14) one gets the following quantal weak conservation laws,

$$Q_{\sigma} = \int_V d^3x \left[\frac{\partial \mathsf{L}_{\text{eff}}}{\partial\varphi_{'0}}\left(b_{\rho} + b_{\rho}^{\nu}\partial_{\nu}\right) - v_{\rho}^{0} + \partial_{\nu}v_{\rho}^{0\nu} - v_{\rho}^{0\nu}\partial_{\nu}\right]\zeta_{\sigma}^{\rho} = \text{const} \tag{4.15}$$

Thus, we see that if the effective action I_{eff} for a gauge-invariant system is invariant under the conressponding transformation, these quantal weak conservation laws coincide with the conservation laws at the quantum level deriving from the global symmetry transformation (Li and Gao, 1999).

In the following sections we shall give some applications of above formulation to the YM fields and CS theories.

5. YANG–MILLS FIELDS

In YM theories, the Lagrangian is gauge-invariant, the Lagrangian without ghosts violates unitarity, the effective Lagrangian in configuration space can be obtained by using the FP trick in the Lorentz gauge through a transformation of the path integral,

$$\mathsf{L}_{\text{eff}} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + \frac{1}{2\alpha_0}\left(\partial^{\mu}A_{\mu}^a\right)^2 - \partial^{\mu}\bar{C}_a D_{\mu b}^a C_b \tag{5.1}$$

where

$$F_{\mu\nu}^a = \partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a + f_{bc}^a A_{\mu}^b A_{\nu}^c \tag{5.2}$$

$$D_{\mu b}^a = \delta_b^a\partial_{\mu} + f_{cb}^a A_{\mu}^c \tag{5.3}$$

and A_{μ}^a are YM fields, f_{bc}^a are structure constants of the gauge group, C_a and $\bar{C}_a$ are odd ghost fields, and α_0 is a parameter.

Now we study the quantal NI for the non-local transformation to conservation laws in YM theories. It is easy to check that the first term and third term in the effective Lagrangian (5.1) are invariant under the following non-local transformation (Li, 1997)

$$\begin{cases} A_{\mu}^{a'}(x) = A_{\mu}^{a}(x) + D_{\mu\sigma}^{a}\varepsilon^{\sigma}(x) & (5.4a) \\ C_{a}^{'}(x) = C_{a}(x) + i(T_{\sigma})_{b}^{a}C_{b}(x)\varepsilon^{\sigma}(x) & (5.4b) \\ \bar{C}_{a}^{'}(x) = \bar{C}^{a}(x) - i\bar{C}_{b}(T_{\sigma})_{b}^{a}\varepsilon^{\sigma}(x) + \frac{i}{\Box}\partial_{\mu}[\bar{C}_{b}(x)(T_{\sigma})_{b}^{a}\partial^{\mu}\varepsilon^{\sigma}(x)] & (5.4c) \end{cases}$$

where T_{σ} are representation matrices of the generators of the gauge group. Equation (4.4c) can be reduced to

$$\bar{C}_{a}^{'}(x) = C_{a}(x) - i\bar{C}_{b}(x)(T_{\sigma})_{b}^{a}\varepsilon^{\sigma}(x) + i\int\{d^{4}y\Delta_{0}(x,y)\partial_{\mu}[\bar{C}_{b}(y)(T_{\sigma})_{b}^{a}\}\partial^{\mu}\varepsilon^{\sigma}(y)] \qquad (5.4c')$$

where

$$\Box\Delta_{0}(x,y) = i\delta^{4}(x-y) \qquad (5.5)$$

The transformation (5.4c′) is a non-local one. Under the transformation (5.4a), (5.4b) and (5.4c′), from the quantal NI (4.10) and the effective Lagrangian (5.1), one obtains

$$\tilde{D}_{\mu\sigma}^{a}\left(\frac{\delta I_{\text{eff}}}{\delta A_{\mu}^{a}(x)}\right) + i(T_{\sigma})_{b}^{a}\frac{\delta I_{\text{eff}}}{\delta C_{a}(x)}C_{b}(x) - i\bar{C}_{b}(x)(T_{\sigma})_{b}^{a}\frac{\delta I_{\text{eff}}}{\delta\bar{C}_{a}(x)} + \int d^{4}y\tilde{N}_{\sigma}^{a}(x)\left[\partial_{\mu}\left(\frac{\partial \mathsf{L}_{\text{eff}}}{\partial\bar{C}'_{\mu}}\right)\Delta_{0}(y,x)\right] = \frac{1}{\alpha_{0}}\tilde{D}_{\sigma\nu}^{a}[\partial^{\nu}(\partial^{\mu}A_{\mu}^{a})] \qquad (5.6)$$

where

$$\tilde{D}_{\sigma\mu}^{a} = -\delta_{\sigma}^{a}\partial_{\mu} + f_{\sigma c}^{a}A_{\mu}^{c} \qquad (5.7)$$

$$N_{\sigma}^{a}(x) = i\partial_{\mu}\left[\bar{C}_{b}(x)(T_{\sigma})_{b}^{a}\partial^{\mu}\right] \qquad (5.8)$$

Under the Lorentz gauge, using the quantal equation of motion, from (5.6), one has

$$\partial^{x_{\mu}}\int d^{4}y\bar{C}_{b}(x)(T_{\sigma})_{b}^{a}\partial_{x_{\mu}}\left[\partial_{y_{\mu}}\left(\frac{\partial \mathsf{L}_{\text{eff}}}{\partial\bar{C}_{a,y_{\mu}}}\right)\Delta_{0}(y,x)\right] = 0 \qquad (5.9)$$

This leads to the conserved quantity at the quantum level

$$Q_{\sigma}' = \int_{V} d^{3}x\int d^{4}y\bar{C}_{b}(x)(T_{\sigma})_{b}^{a}\partial_{x_{0}}\left[\partial_{y_{\mu}}\left(\frac{\partial \mathsf{L}_{\text{eff}}}{\partial C_{,y_{\mu}}^{a}}\right)\Delta_{0}(y,x)\right] = \text{const} \qquad (5.10)$$

Substituting (5.1) into (5.10), one get

$$Q'_\sigma = \int_V \int d^3x d^4y \bar{C}_b(x)(T_\sigma)^a_b \left(\partial_\nu D^{a\nu}_e C_e\right) \partial_{x_0} \Delta_0(y, x) = \text{const} \tag{5.11}$$

A system with a gauge-invariant Lagrangian is a constrained Hamillonian system, the path-integral quantization of this system can be formulated by using FS scheme (Faddeev, 1970; Senjanovic, 1976). In the coulomb gauge the phase-space generating functional of Green function for YM field can also be written as (the theory is gauge independent) (Li, 1994c)

$$Z[J] = \int \mathrm{D} A^a_\mu \mathrm{D}\pi^\mu_a \mathrm{D}\, \bar{\mathrm{C}}^a\, \mathrm{D}C^a \mathrm{D}\lambda^a_k$$

$$\exp\left\{ i \int d^4x \left[\mathrm{L}^P_{\text{eff}} + J^\mu_a A^a_\mu + \bar{C}^a J_a + \bar{J}_a C^a + J^a_k \lambda^a_k \right] \right\} \tag{5.12}$$

where

$$\mathrm{L}^P_{\text{eff}} = \mathrm{L}^p + \mathrm{L}_m + \mathrm{L}_{gh} \tag{5.13}$$

$$\mathrm{L}^P = \pi^\mu_a \dot{A}^a_\mu - \mathrm{H}_C \tag{5.14}$$

$$\mathrm{L}_m = \lambda^a_k \Lambda^a_k - \frac{1}{2\alpha_k} \left(\Omega^a_k\right)^2 \qquad (k = 1, 2) \tag{5.15}$$

$$\mathrm{L}_{gh} = -\partial^\mu \bar{C}^a \mathrm{D}^a_{\mu b} C^b \tag{5.16}$$

where Λ^a_k and Ω^a_k are constraints and gauge conditions. It is easy to check that $\mathrm{L}^p$and L_{gh} are invariant under transformation (5.4). We use $\varepsilon^\sigma(x) = \varepsilon^\nu A^\sigma_\nu(x)$ in the transformation (5.4), where $\varepsilon^\nu$are numerical parameters. Since the variations of the first-class constraints under the gauge transformation (5.4a) are within the constraint hypersurface (Li, 1993c), thus, $\delta \mathrm{L}_m \approx 0$ under the transformation (5.4). Therefore, $\delta I^p_{\text{eff}} \approx 0$ under the transformation (5.4), where the sign$\approx$ means equality on the constraint hypersurface (including gauge constraints). Using the quantal canonical equations, from (3,8), one obtains the quantal conserved quantities

$$Q_v = \int d^3x \left\{ \pi^\mu_a D^a_{\mu\sigma} A^\sigma_\nu + i\pi_a (T_\sigma)^a_b C^b A^\sigma_\nu - i\bar{\pi}_a \bar{C}^b (T_\sigma)^a_b A^\sigma_\nu \right.$$

$$\left. + \bar{\pi}_a \int d^4y \Delta_0(x, y) \partial_\mu [\bar{C}^b(y)(T_\sigma)^a_b \partial^\mu A^\sigma_\nu(y)] \right\} \tag{5.17}$$

where π_a and $\bar{\pi}_a$ are canonical momenta with respect to C^a and $\bar{C}^a$ respectively.

Thus, we have shown that for certain cases by using the quantal equations of motion the quantal NI (or strong conservation laws) may be converted into the weak conservation quantities even if the effective Lagrangian is not invariant under the specific transformation. This algorithm to deduce the quantal conservation laws is different from quantal first Noether theorem (Li, 1997; Li and Gao, 1999).

6. NON-ABELIAN CHERN-SIMONS THEORIES

A lot of recent work on (2 + 1)-dimensional Chern–Simons (CS) gauge theories revealed the occurrence of fractional spin and statistics (Banerjee, 1994; Kim *et al.*, 1994; Li, 1999). This may be related to the fractional quantun Hall effect and T_c-superconductivity (Lerda, 1992). However, in the present study of CS theories coupled to the matter fields, some basic problems need clarifying. First, in the Hamiltonian analysis of the models, the gauge field was eliminated by using classical equations of motion and gauge conditions, but the constraints associated with the gauge field are unaccounted. Hence the question, is this result equivalent to the original model at the quantum level (Banerjee and Chakraborty, 1994)? Second, in the discussion of the angular momenta for anyons, the results were deduced by using classical Noether theorem (Banerjee, 1994; Kim *et al.*, 1994). Whether they are valid at the quantum level. Third, some authors have putted forward that whether the properties of angular momenta for anyons still survive in the Maxwell–Chern–Simons theories (Kim *et al.*, 1994), which is need further study.

Let us now consider the (2 + 1)-dimensional non-Abelian CS term coupled to the scalar field with the Maxwell term whose Lagrangian is given by

$$\mathsf{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + (D_\mu\varphi)^+(D^\mu\varphi) + \frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}\left(\partial_\mu A^a_\nu A^a_\rho + \frac{1}{3}f^a_{bc}A^a_\mu A^b_\nu A^c_\rho\right) \tag{6.1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\mu - \partial_\nu A^a_\mu + f^a_{bc}A^b_\mu A^c_\mu \tag{6.2}$$

and φ is an N-component scale field, $D_\mu = \partial_\mu - iT^aA^a_\mu$, T^a are generator of gauge group, $[T^a, T^b] = if^a_{bc}T^c$, $tr(T^aT^b) = \frac{1}{2}\delta^{ab}$, The gauge invariance of non-Abelian CS term requires the parameter $\kappa = \frac{n}{4\pi}$ $(n \in Z)$ (Deser *et al.*, 1982).

First of all we formulate the path-integral quantization for this model, and then the quantal canonical symmetries will be further investigated. The canonical momenta π^μ_a, π^+ and π associated with A^a_μ, φ and φ^+ are given by

$$\pi^\mu_a = F^{\mu 0}_a + \frac{\kappa}{4\pi}\varepsilon^{0\mu\nu}A^a_\nu \tag{6.3}$$

$$\pi^+ = (D_0\varphi)^+ \tag{6.4}$$

$$\pi = D_0\varphi \tag{6.5}$$

The constraints are

$$\Lambda^a_1 = \pi^0_a \approx 0 \tag{6.6}$$

$$\Lambda^a_2 = D_i\pi^i_a + \frac{\kappa}{4\pi}\varepsilon^{ij}\partial_i A^a_j \approx 0 \tag{6.7}$$

where the convention $\varepsilon^{012} = \varepsilon^{12} = 1$ is used. It is easy to check that the Λ_1^a and Λ_2^a are first-class constraints. According to the rule of path integral quantization of constrained Hamiltonian system, for each first-class constraint, a corresponding gauge conditions should be chosen. Consider the Coulomb gauge,

$$\Omega_2^a = \partial^i A_i^a \approx 0 \tag{6.8}$$

The consistency requirement of this gauge constraint implies another gauge conditions

$$\Omega_1^a = \partial_i \pi_i^a + \nabla^2 A_0^a - f_{bc}^a A_i^b \partial^i A_0^c \approx 0 \tag{6.9}$$

One can find that $\det|\{\Lambda^a, \Omega^b\}| = \det M_c^{ab}$, where

$$M_c^{ab} = \left(\delta^{ab}\nabla^2 - f_{bc}^a A_i^c \partial^i\right)\delta(x-y) \tag{6.10}$$

The factor $\delta(\partial^i A_i^a)\det M_c^{ab}$ can be replaced by $\delta(\partial^\mu A_\mu^a)\det M_c^{ab}$ (Foussats *et al.*, 1995, 1996; Sundermeyer, 1982), where

$$M_c^{ab} = (\delta^{ab}\partial^2 - f_{bc}^a A_\mu^c \partial^\mu)\delta(x-y) \tag{6.11}$$

Thus, the phase-space generating functional of the Green function for this model can be written as (Li, 1997; Li and Long, 1999)

$$Z[J] = \int \mathrm{D}A_\mu^a\, \mathrm{D}\pi_a^\mu\, \mathrm{D}\varphi\, \mathrm{D}\pi\, \mathrm{D}\varphi^+\, \mathrm{D}\pi^+\, \mathrm{D}\lambda\, \mathrm{D}\bar{C}^a \mathrm{D}C^a$$
$$\exp\left\{i\int d^3x\left(\mathrm{L}_{\mathrm{eff}}^P + J_a^\mu A_\mu^a + J^+\varphi + \varphi^+ J + \bar{J}_a C^a + \bar{C}^a J_a\right)\right\} \tag{6.12}$$

where J_a^μ, J^+, J, $\bar{J}_a$ and J_a are exterior sources with respect to A_μ^a, φ, φ^+, C^a and $\bar{C}^a$ respectively, and

$$\mathrm{L}_{\mathrm{eff}}^P = \mathrm{L}^P + \mathrm{L}_g + \mathrm{L}_{gh} + \mathrm{L}_m \tag{6.13}$$

$$\mathrm{L}^P = \pi_a^\mu \dot{A}_\mu^a + \pi^+\dot{\varphi} + \dot{\varphi}^+\pi - \mathrm{H}_C \tag{6.14}$$

$$\mathrm{L}_g = -\frac{1}{2\alpha_2}\left(\partial^\mu A_\mu^a\right)^2 \tag{6.15}$$

$$\mathrm{L}_{gh} = -\partial^\mu \bar{C}^a D_{\mu b} C^b \quad \left(D_{\mu b}^a = \delta_b^a \partial_\mu - f_{bc}^a A_\mu^c\right) \tag{6.16}$$

$$\mathrm{L}_m = \lambda_1^a \Lambda_1^a + \lambda_2^a \Lambda_2^a - \frac{1}{2\alpha_1}\left(\Omega_1^a\right)^2 \tag{6.17}$$

and H_C is a canonical Hamiltonian density.

Let us consider BRS (Beechi–Rouet–Stora) transformation:

$$\delta\varphi = -i\tau T^a C^a \varphi, \quad \delta\varphi^+ = i\tau\varphi^+ T^a C^a \tag{6.18a}$$

$$\delta C^a = \frac{1}{2} f^a_{bc} C^b C^c, \quad \delta \bar{C}^a = -\frac{1}{\alpha_2} \partial^\mu A^a_\mu \tag{6.18b}$$

$$\delta A^a_\mu = \tau D^a_{\mu b} C^b \tag{6.18c}$$

where τ is a Grassmann parameter. The action connected with the term $\mathsf{L}^P + \mathsf{L}_g + \mathsf{L}_{gh}$ is invariant under the BRS transformation (5.18) at the quantum level. The variations of the first-class constraints under the gauge transformation (6.18c) are within the constraint hypersurface (Li, 1995c). Thus $\delta\,\mathsf{L}_m \approx 0$ under the transformation (6.18). Therefore, $\delta I^P_{\text{eff}} \approx 0$ under the transformation (6.18). From (3.8) one gets weak conserved BRS quantity at the quantum level.

$$Q_B = \int d^2x \left(\pi^\mu_a \delta A^a_\mu + \pi^+ \delta\varphi + \delta\varphi^+ \pi + \bar{R}_a \delta C^a + \delta \bar{C}^a R_a \right) \tag{6.19}$$

where $\bar{R}_a$ and R_a are canonical monenta conjugate to C_a and $\bar{C}^a$ respectively.

If we only consider the transformation of A^a_μ, φ and φ^+, fixing the ghost fields,

$$\begin{aligned} \delta\varphi &= -i\tau T^a C^a \varphi, \quad \delta\varphi^+ = i\tau\varphi^+ T^a C^a \\ \delta A^a_\mu &= \tau D^a_{\mu b} C^b, \quad \delta C^a = \delta \bar{C}^a = 0 \end{aligned} \tag{6.20}$$

under the transformation (6.20), the change of L^P_{eff} is given

$$\delta\,\mathsf{L}^P_{\text{eff}} = V_a \varepsilon^a(x) = F_a \varepsilon^a(x) + f^a_{bc} \partial^\mu \bar{C}^a \partial_\mu \varepsilon^c(x) \tag{6.21}$$

within the constraint hypersurface, where $\varepsilon^a(x) = \tau C^a(x)$, and F_a do not contain the derivatives of the $\varepsilon^a(x)$. From (3.8), one obtains weak conserved PBRS quantity at the quantum level (P stands for "partial")

$$Q = \int d^2x \left(\pi^\mu_a \delta A^a_\mu + \pi^+ \delta\varphi + \delta\varphi^+ \pi - f^a_{bc} \dot{\bar{C}}^a C^b C^c \right) \tag{6.22}$$

This quantal conserved quantity Q differs from Q_B in (6.19).

The above conserved quantity Q_B and Q can also be derived by using the configuration-space generating functional as performed in Section 5.

As is well known, BRS charge annihilates vacuum state, this conserved PBRS charge may also impose some supplementary conditions on physical states as well as BRS charge charge and ghost charge. Work along this line is in process.

The effective canonical action is also invariant under the spatial rotation transformation in the (x_1, x_2) plan, one can obtain the conserved angular momentum for non-Abelian CS theories at the quantum level.

$$J = \int d^2x \left[\pi^\mu_a \left(x_2 \frac{\partial A^a_\mu}{\partial x_1} - x_1 \frac{\partial A^a_\mu}{\partial x_2} \right) + \pi^\mu_a \left(\sum\nolimits_{12}{}^{\mu\nu} \right) \right] A^a_\nu$$

$$
\begin{aligned}
&+\pi^+\left(x_2\frac{\partial\varphi}{\partial x_1}-x_1\frac{\partial\varphi}{\partial x_2}\right)+\left(x_2\frac{\partial\varphi^+}{\partial x_1}-x_1\frac{\partial\varphi^+}{\partial x_2}\right)\pi\\
&+\bar{R}_a\left(x_2\frac{\partial\bar{C}^a}{\partial x_1}-x_1\frac{\partial\bar{C}^a}{\partial x_2}\right)+\left(x_2\frac{\partial\bar{C}^a}{\partial x_1}-x_1\frac{\partial\bar{C}^a}{\partial x_2}\right)R_a \qquad (6.23)
\end{aligned}
$$

where $(\sum_{jk})_{\mu\nu} = g_{j\mu}g_{k\nu} - g_{j\nu}g_{k\mu}$. Thus, we see that the quantal conserved angular momentum in this model differs from classical Noether one in that one needs to take into account the contribution of angular momentum of ghost fields in Maxwell-non-Abelian CS theory. We do not think the conclusions in classical theories are always validity in quantum theories (Antillon *et al.*, 1995; Banergee and Chakraborty, 1996). It had been pointed out that in some Abelian CS models where is no ghost field in quantized effective Lagrangian, and the fractional spin properties are preserved at the quantum level (Banerjee, 1994; Kim *et al.*, 1994). The property of fractional spin in non-Abelian CS theories needs further study in quantum theories.

7. CONCLUSIONS AND DISCUSSION

Classical NI refers to the invariance of an action integral of the system under the local transformation. Here we study the quantal local and non-local symmetries for a system with a regular/singular Lagrangian. The path integrals provide a useful tool. In the theory of path integral quantization for a dynamical system, the phase-space path integrals are more fundamental than configuration-space path integrals. Based on the phase-space generating functional of the Green function for a system with a regular/singular Lagrangian, the quantal canonical NI under the local and non-local transformation in extended phase space have been derived, respectively. These identities hold true no matter whether the Jacobian of the corresponding transformation is equal to unity or not. For a system with a regular Lagrangian, the expressions of quantal canonical NI coincide with the classical ones, but for a singular Lagrangian one must use an effective canonical action I_{eff}^P instead of canonical action I^P in the corresponding expressions. For a gauge-invariant system a simpler and more useful quantization scheme is FP trick, from which the configuration-space generating functional of the Green function can be formulated. Based on this generating functional, the quantal NI under the local and non-local transformation in configuration space for gauge-invariant system have been also deduced. These identities also hold true whether the Jacobian of the transformation is equal to unity or not. It had been shown that in a certain case, the quantal NI may convert to quantal strong and weak conservation laws, this algorithm to derive quantal (weak) conserved quantities is different from quantal first Noether theorem. We give some preliminary applications of above results to YM fields. The quantal conserved quantities for local transformation are also found. The application of above formulation to non-Abelian CS term coupled to the scalar field is also given, the quantal conserved BRS and PBRS quantities are obtained. The quantal conserved angular momentum for non-Abelian Maxwell CS theory is found, which differs from classical Noether one in that one needs to take into account the contribution of angular momentum of ghost fields. But in the Abelian CS theories there is no ghost field, the angular momentum at the quantum level coincides with classical Noether one. The property of fractional spin is preserved at the quantum level in the Abelian CS theories (Banerjee, 1994; Kim *et al.*, 1994; Li, 1996).

It had been pointed out that the anomalies can be viewed as a result of the non-invariance of the functional measure under the symmetry transformation (Fujikawa, 1980, 1981). The result (6.23) indicates that the anomalies may appear in a case with the invariance of the functional measure under a symmetry transformation.

The conserved angular momentum (6.23) is not gauge invariant as in the Abelian CS theories (Banerjee, 1994; Kim *et al.*, 1994). Those angular momentum had been constructed from the symmetric energy-momentum tensor for Abelian CS theories (Banerjee, 1993, 1994; Kim *et al.*, 1994) in order to preserve those gauge invariant. For non-Abelian CS theories, we can consider a gauge-translation transformation

$$\begin{aligned}\phi'(x+\varepsilon) &= \exp\{ig\varepsilon^\nu A_\nu(x)\}\phi(x)\\ A'_\mu(x+\varepsilon) &= \exp\{ig\varepsilon^\nu A_\nu(x)\}A_\mu(x)\exp\{-ig\varepsilon^\nu A_\nu(x)\}\\ &\quad -\frac{i}{g}\partial_\mu \exp\{ig\varepsilon^\nu A_\nu(x)\}\exp\{-ig\varepsilon^\nu A_\nu(x)\}\end{aligned} \tag{7.1}$$

and can derive the gauge-invariant energy-momentum tensor $T_{\mu\nu}$, where ϕ stands for φ, φ^+, C^a and $\bar{C}^a$, and A_μ are non-Abelian CS gauge fields, and ε^ν are parameters. The coefficient κ connected with CS term also appears in the expression of the angular momentum $J = \int d^2x\varepsilon^{ij}x_i T_{0j}$, the property of fractional spin can be further study, and work along these lines is in progress.

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